HAMILTONIAN CIRCUITS IN SIMPLE 3-POLYTOPES WITH UP TO 26 VERTICES

BY

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ABSTRACT

Using a theorem of Butler and Goodey, and using several new reductions, we show that every simple 3-polytope with 26 or fewer vertices has a Hamiltonian circuit.

1. Introduction

The problem, posed by V. Klee as Problem 20 in [1], is to determine the maximum number n such that every simple 3-polytope with fewer than n vertices is Hamiltonian. Examples by Lederberg, Bosak and Barnette show $n \leqq 38$, Lederberg [3] proved $n \geqq 20$ and recently J. Butler [7] and P. R. Goodey [5] proved independently $n \geqq 24$. For further historical details, for related problems and for basic notations compare B. Grünbaum [2] and [4]. The purpose of this note is to prove:

THEOREM 1. *Every 3-polytope with fewer than 28 vertices has a Hamiltonian circuit.*

2. Definitions and Preliminaries

The graphs we deal with will be the graphs of simple 3-polytopes. According to Steinitz's theorem (see [2, Ch. 13.1]) these graphs are characterized as those that are planar, 3-connected and 3-valent. An embedding of such a graph into the plane π subdivides π into faces which correspond to the facets of the polytope. We shall use the following lemma which is easily verified.

LEMMA 1. *If no two faces of a planar 2-connected graph have a multiply-connected union then the graph is 3-connected.*

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Let G be the graph of a simple 3-polytope. An *n-cut* of G consists of a set C of n edges of G such that the removal of these edges separates G into two connected components and no proper subset of C has this property. The components separated by an n-cut are called *n-pieces*. An n-cut is *trivial* provided one of its n-pieces contains no circuit. Likewise an n-piece containing no circuit is called *trivial*.

For 3-valent graphs, 3-connectedness is equivalent to 3-edge-connectedness; thus G has no k-cuts with $k < 3$. G is called *cyclically-n-connected* (abbreviated cn-connected) if and only if there is no k-cut ($k < n$) such that both k-pieces contain circuits. That is, G is cn-connected if and only if any k-cut of G with $k < n$ is trivial. Of course, every 3-polyhedral graph is c3-connected. Finally G is called c*n-connected if and only if there is no k-cut ($k < n$) such that both k-pieces contain more than one circuit.

Concerning Hamiltonian circuits, we have from Butler [7] and Goodey [5] the following.

THEOREM 2. (a) *In any simple 3-polytope with less than 24 vertices each edge is used by some Hamiltonian circuit.*

(b) *Let G be a minimal non-Hamiltonian simple 3-polytope. If G is not c4-connected, then G has at least 38 vertices.*

In what follows, G will always refer to a simple 3-polyhedral non--Hamiltonian graph with a minimum number v of vertices and we assume $v < 28$. We shall show that this is impossible, and in view of Theorem 2, we have to examine only c4-connected simple 3-polytopes with 24 or 26 vertices.

3. Substitutions

Our main tool will be the fact that certain n-pieces cannot occur as n-pieces in G.

LEMMA 2. *G cannot contain adjacent quadrilaterals.*

PROOF. Assume that G contains adjacent quadrilaterals. We replace this 4-piece C by a trivial 4-piece C' (an edge) as indicated in Figure 1, producing a new graph G'. By Lemma 1, G' will be 3-connected unless the faces α and β meet on an edge e. But if this is the case then a, d, and e form a 3-cut, and so do b, c, and e. Since G is c4-connected, these 3-cuts must be trivial. This implies that G is the graph of the cube which is Hamiltonian.

Since G' is c3-connected and has fewer vertices than G, it has a Hamiltonian circuit. But any path or pair of paths through C' in G' can be extended to a similar path or pair of paths through C in G, thus G also has a Hamiltonian circuit.

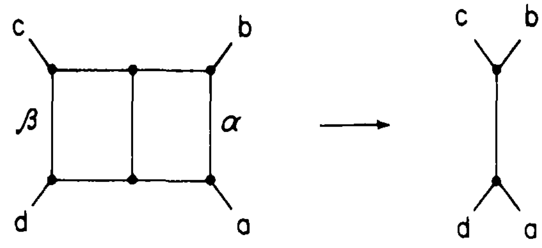

Fig. 1.

Lemma 3. *G cannot contain a quadrilateral adjacent to a pentagon.*

Proof. We use again a substitution, as indicated in Figure 2, thus obtaining a graph G' with fewer than 24 vertices. If G' is not 3-connected, then by Lemma 1, faces α and β (see Fig. 2) or faces α and γ meet on an edge. Suppose without loss of generality, α and β meet on an edge e. Now a, b, and e form a 3-cut which must be trivial; this however implies that δ is a quadrilateral, contradicting Lemma 2. Now, G' is Hamiltonian, furthermore it has a Hamiltonian circuit using any prescribed edge (according to Theorem 2 (a)). Thus G is also Hamiltonian, since any Hamiltonian circuit in G' using the edge marked by an asterisk extends to a Hamiltonian circuit in G.

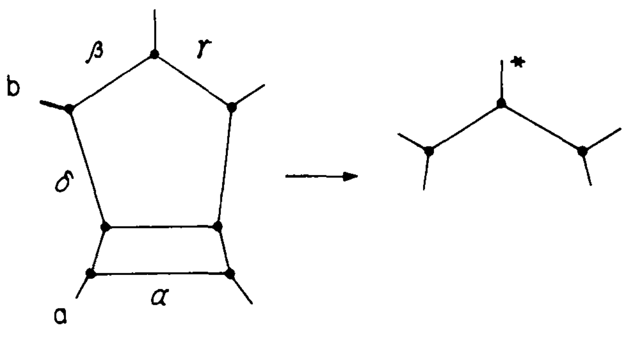

Fig. 2

Lemma 4. *G is* c*4-*connected.*

Proof If G were not c*4-connected, there would exist a 4-cut yielding 4-pieces with more than 4 vertices. Because $v < 28$, at least one of the 4-pieces would have less than 14 vertices. Yet by inspection one finds that any 4-piece without triangles with more than 4 and less than 14 vertices contain a quadrilateral adjacent to another quadrilateral or to a pentagon and is therefore excluded by Lemma 2 or 3.

LEMMA 5. *G cannot contain a quadrilateral adjacent to a hexagon.*

The proof uses the substitution indicated in Figure 3 and is completely analoguous to that one of Lemma 3. To ensure 3-connectedness of G' one needs additionally that α and β are not adjacent; if this were the case, G would have a 4-cut yielding 4-pieces each with more than one circuit, contrary to Lemma 4.

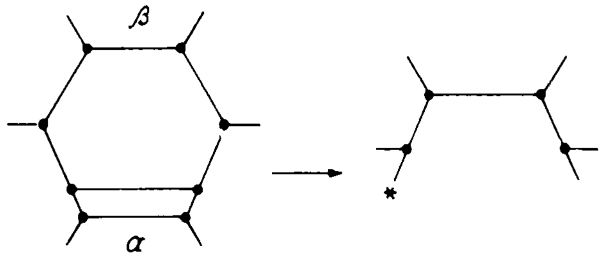

Fig. 3.

4. Proof of Theorem 1

Let p_n denote the number of n-gons of G. Since G is c4-connected, $p_3 = 0$. As we shall see, any possible value of p_4 will yield a contradiction to $v < 28$. From Euler's formula (see [2, Ch. 13.3]) we have

$$2p_4 + p_5 = 12 + \sum_{k>6}(k-6)p_k \tag{1}$$

and the number v of vertices is given by

$$v = \tfrac{1}{3}\sum kp_k. \tag{2}$$

Case I. $p_4 = 0$. Since G is c*4-connected, $p_4 = 0$ implies that G is c5-connected. The c5-connected graphs with fewer than 28 vertices have been shown to be Hamiltonian (see [6]).

Case II. $p_4 = 1$. Because of Lemmas 2, 3, and 5, the quadrilateral is surrounded by n-gons with $n \geqq 7$. Thus we obtain from (1), $p_5 \geqq 16 - 2p_4 = 14$ and then from (2),

$$v \geqq \tfrac{1}{3}(4 + 14\cdot 5 + 4\cdot 7) > 28.$$

CASE III. $2 \leqq p_4 \leqq 6$. Already at least 6 n-gons with $n \geqq 7$ are needed to surround two quadrilaterals. Thus with (1), $p_5 \geqq 18 - 2p_4$ and with (2),

$$v \geqq \tfrac{1}{3}(4p_4 + 5p_5 + 42) \geqq \tfrac{1}{3}(132 - 4\,6p_4) > 28.$$

Finally $p_4 \geqq 7$ cannot occur since, according to Lemma 2, quadrilaterals are disjoint which implies $v \geqq 4p_4$.

This completes the proof of Theorem 1.

5. Remarks

It should be noted that Lemma 2 holds also if G is a minimal, non-Hamiltonian, c4-connected simple 3-polytope with fewer than 42 vertices. (It is conjectured, compare for instance Faulkner and Younger [6], that there exists none and that 42 is the minimum number of vertices). The proof is quite the same using additionally Theorem 2(b).

There exist a lot of substitutions similar to that of Lemma 2 which are not restricted to the case $v < 28$. Two simple examples are shown in Figure 4. Using such substitutions, Lemma 4 may be extended at least up to 30 vertices.

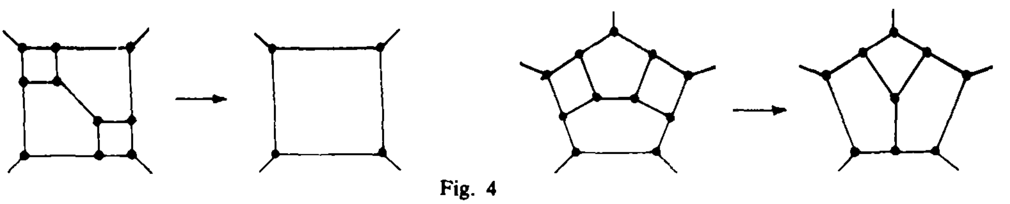

Fig. 4

The authors originally proved Theorem 1 independently. Upon combining the two results, Wegner was able to arrive at the short proof found in this paper.

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